Cosmic no-hair: non-linear asymptotic stability of de Sitter universe

Marco Bruni[♮], Filipe C. Mena[♭], & Reza Tavakol[♭]
[♮]*School of Computer Science and Mathematics, Portsmouth University, Portsmouth PO1 2EG, U.K.*
[♭]*Astronomy Unit, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road London E1 4NS, U.K.*

Abstract

We study the asymptotic stability of de Sitter spacetime with respect to non-linear perturbations, by considering second order perturbations of a flat Robertson-Walker universe with dust and a positive cosmological constant. Using the synchronous comoving gauge we find that, as in the case of linear perturbations, the non-linear perturbations also tend to constants, asymptotically in time. Analysing curvature and other spacetime invariants we show, however, that these quantities asymptotically tend to their de Sitter values, thus demonstrating that the geometry is indeed locally asymptotically de Sitter, despite the fact that matter inhomogeneities tend to constants in time. Our results support the inflationary picture of frozen amplitude matter perturbations that are stretched outside the horizon, and demonstrate the validity of the cosmic no-hair conjecture in the nonlinear inhomogeneous settings considered here.

PACS numbers: 98.80.Hw, 98.80.Cq, 04.25.Nx

The cosmic no-hair conjecture [1] states – roughly speaking – that 'all expanding universe models with a positive cosmological constant asymptotically approach the de Sitter solution' [2]. Previous studies of this conjecture fall into two main groups: those involving special models, e.g. containing Killing vectors or special Petrov types, and those using a linear perturbative analysis. Among the first group is the work of Wald [2], who proved the conjecture for ever expanding homogeneous models, as well as studies of inhomogeneous models [3]. Among the second group are studies of the stability of de Sitter with respect to linear tensor (gravitational wave) perturbations [4], as well as investigations involving both linear scalar and tensor modes [5,6], where the scalar modes arise in presence of matter. The authors of [5,6] studied the dynamics of linear perturbations in Friedmann-Lemaître-Robertson-Walker (FLRW) models with dust and a positive cosmological constant Λ, by employing a synchronous comoving gauge, and showed that both scalar and tensor modes tend to constants in time, while the vector modes were shown to decay [5]. This establishes the stability of de Sitter in these linear perturbative settings, but not its asymptotic stability [5]. It was also shown in [6], however, that there exists a coordinate transformation (valid within the horizon, see below) in terms of which the asymptotic metric becomes locally de Sitter in static coordinates [7]. This at least partly demonstrates that one in fact has a local asymptotic stability, in the sense that for a freely falling observer in such a frame the spacetime approaches de Sitter in time, exponentially fast.

A number of studies have addressed this conjecture from a broader perspective (see e.g. [8] and Refs. therein), and in particular linear perturbations of de Sitter have been studied extensively in the context of the inflationary scenario (see e.g. [9,10] and Refs. therein). The latter has been one of the main motivations for the interest in the cosmic no-hair conjecture. Indeed inflation was devised as a mechanism to account for the observed large-scale homogeneity and isotropy of the universe, and while it is natural to assume a FLRW model to work out observational predictions of inflationary models such as perturbation spectra, it is nevertheless important to address the fundamental question of whether locally a FLRW geometry can arise from an open set of generic initial data, whose memory is lost through an inflationary phase.

The aims of this Letter are twofold: to extend the perturbative analysis of the stability of de Sitter to second order and to provide an invariant characterization of the asymptotic results of both first and second order perturbations.

Firstly, in order to make contact with earlier results, we study the asymptotic evolution of the first and second order perturbations of a flat FLRW model with dust plus Λ. Asymptotically, this can be viewed as a de Sitter universe with dust perturbations. Secondly, to throw more light on the asymptotic state of the perturbed universe model, we employ curvature and other spacetime invariants. This Letter contains a short account of our results; a more detailed analysis will be presented elsewhere [11].

We proceed by using the formalism of [12,13], extending it to include Λ. In particular we work in the comoving synchronous gauge and assume vanishing vorticity. There is no loss of generality in the latter choice, as vorticity decays during expansion. The line element of the perturbed spacetime can be written as

$$ds^2 = a^2(\tau)(-d\tau^2 + \gamma_{\alpha\beta}dx^\alpha dx^\beta) \ , \tag{1}$$

where a is the scale factor, τ is the conformal time given by $d\tau = dt/a(t)$ and, in usual

notation

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} + \gamma^{(1)}_{\alpha\beta} + \frac{1}{2}\gamma^{(2)}_{\alpha\beta} \ . \tag{2}$$

The full equations to be considered are: those for the evolution of the expansion tensor $\theta_{\alpha\beta}$

$$\theta'_{\alpha\beta} + \theta\theta_{\alpha\beta} + R^*_{\alpha\beta} = \frac{1}{2}\rho\delta_{\alpha\beta} + \Lambda\delta_{\alpha\beta} \ , \tag{3}$$

its trace, the Raychaudhuri equation

$$\theta' + \theta^{\alpha\beta}\theta_{\alpha\beta} + \frac{1}{2}\rho = \Lambda \ , \tag{4}$$

the continuity equation

$$\rho' + \theta\rho = 0 \ , \tag{5}$$

and the energy constraint equation

$$\theta^2 - \theta^{\alpha\beta}\theta_{\alpha\beta} + R^* = 2(\rho + \Lambda) \ , \tag{6}$$

where ρ is the matter-density, $\theta^\alpha_\beta = u^\alpha_{;\beta} = \frac{1}{2}\gamma^{\alpha\delta}\gamma'_{\delta\beta}$, R^* is the 3-Ricci scalar and the prime denotes the derivative with respect to τ. Since we are interested in the asymptotic behavior of the perturbations, we shall assume that the background evolution has already reached the asymptotic regime, with the Λ term dominating in the Friedmann equation. The Friedmann equation can in this case be solved to give

$$a(\tau) = -\frac{H^{-1}}{\tau} \ , \qquad H = \sqrt{\frac{\Lambda}{3}} \ , \tag{7}$$

where the conformal time τ is negative and tends to zero as a goes to infinity.

We start by considering the first order perturbations since they act as source terms in the second order perturbation equations. In the linear theory the scalar and tensor modes are independent and for an irrotational spacetime one can set the vector modes equal to zero. The evolution equation for the scalar perturbations is obtained by combining (3), (5) and (6) and can be written in terms of the density contrast $\delta = (\rho - \rho_b)/\bar{\rho}$ in the form

$$\delta'' + \frac{a'}{a}\delta' - \frac{1}{2}a^2\bar{\rho}\delta = 0 \ , \tag{8}$$

where $\bar{\rho} = \rho_0/a^3$ is the background density and ρ_0 is a constant. Note that the last term of this equation is asymptotically negligible and therefore the asymptotic solution of (8) has two modes, one constant and the other decaying. Expanding the full solution of this equation around $\tau = 0$ one obtains up to second order in τ

$$\delta(\tau, \mathbf{x}) \approx C_1(\mathbf{x}) \left(\frac{\Gamma(\frac{2}{3})}{A^{\frac{2}{3}}\pi} + \frac{1}{6}\frac{3^{\frac{1}{6}}A^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}\tau^2 \right) + C_2(\mathbf{x})\tau^2 \ , \tag{9}$$

where $A^2 = \frac{1}{2}\rho_0 H$; C_1 and C_2 are arbitrary spatial functions related to the initial data. The scalar metric perturbations are related to δ [12] and also tend to constants in time [11].

The tensor perturbation $\pi_{\alpha\beta}$ is the transverse traceless part of $\gamma^{(1)}_{\alpha\beta}$ and is gauge-invariant. The asymptotic form of the equation for $\pi_{\alpha\beta}$ is similarly obtained from the linearization of the evolution equations, and for each Fourier mode k is given by

$$\pi''_{\alpha\beta} - \frac{2}{\tau}\pi'_{\alpha\beta} + k^2\pi_{\alpha\beta} = 0 \; . \tag{10}$$

This equation can be solved to give a solution for each mode k

$$\pi_{\alpha\beta} = a_{\alpha\beta}(\sin k\tau - k\tau\cos k\tau) + b_{\alpha\beta}(\cos k\tau + k\tau \sin k\tau) \; , \tag{11}$$

where $a_{\alpha\beta} = a_{\alpha\beta}(k)$ and $b_{\alpha\beta} = b_{\alpha\beta}(k)$. A formal solution $\pi_{\alpha\beta}(\mathbf{x},\tau)$ can then be obtained in terms of a Fourier integral, which can asymptotically be approximated by

$$\pi_{\alpha\beta} \approx \int_0^\infty \left(b_{\alpha\beta}\left(1 + \frac{k^2\tau^2}{2}\right) + \frac{a_{\alpha\beta}k^3\tau^3}{3}\right) e^{-i\mathbf{kx}} d\mathbf{k} \; . \tag{12}$$

Thus both the first order scalar and tensor perturbations (9) and (11) asymptotically approach constants in time.

The analysis so far confirms known results [4–6] and sets the stage for proceeding to the second order. Before doing so, it is however worthwhile taking a brief look at some first order gauge-invariant quantities. The Electric Weyl tensor satisfies [14]

$$a^{(3)}\nabla_b E^b_a = \tfrac{1}{3}\bar{\rho}\mathcal{D}_a \quad \Leftrightarrow \quad 2\tfrac{k^2}{a^2}\Phi_H = \bar{\rho}\varepsilon_m \; , \tag{13}$$

where the equivalence with the equation on the right hand side (Eq. (4.3) in Bardeen [15]) is in Fourier space; $\mathcal{D}_a$ is the comoving fractional density gradient of density, a covariant gauge-invariant measure of inhomogeneity [16], and ε_m is the equivalent Bardeen variable [14]. Now both these quantities evolve exactly as the density perturbation in the comoving gauge above, and thus tend to constant values in time in the present context. The simple Eq. (13) is the relativistic perturbation equivalent of the Poisson equation, and Bardeen's potential Φ_H is the gauge-invariant analogue of the Newton's potential [14]. This equation carries a great deal of useful information. Firstly it shows that in the absence of matter, the Electric Weyl tensor is transverse, so that there are no scalar modes in this case, $\Phi_H = 0$, but only gravitational waves (cf. [17]. Secondly it shows that Φ_H decays as $\sim a^{-1}$ when the density perturbation is frozen, with a corresponding decay in (the scalar part of) E^b_a. Finally Φ_H directly corresponds to the metric perturbation variable in the longitudinal gauge [9], which therefore also decays (cf. [12]).

Concerning the second order perturbations, the spatial metric in (1) can be written as

$$\gamma^{(2)}_{\alpha\beta} = -2\phi^{(2)}\delta_{\alpha\beta} + \chi^{(2)}_{\alpha\beta} \, , \tag{14}$$

where $\phi^{(2)}$ and $\chi^{(2)}_{\alpha\beta}$ denote respectively the trace and trace-free parts. The usual procedure is then to substitute (14) and (7) in the field equations, using a similar procedure as in the linear case. Keeping the lowest order terms of (9) and (12) on the right hand side of the Raychaudhuri equation we obtain

$$\phi^{(2)\prime\prime} - \frac{1}{\tau}\phi^{(2)\prime} + \frac{1}{2}\sqrt{\frac{\Lambda}{3}}\rho_0\tau\phi^{(2)} = F(\mathbf{x})\tau + O(\tau^2) \ , \tag{15}$$

where the source term F is a spatial function (given in [11]) that depends quadratically on first order perturbations, and in particular from the four initial data quantities $C_1, C_2, a_{\alpha\beta}$ and $b_{\alpha\beta}$. The solutions to (15) have the form

$$\phi^{(2)} = \frac{F(\mathbf{x})}{A^2} + \tau C_3(\mathbf{x}) J_{\frac{2}{3}}\left(\tfrac{2A\tau^{3/2}}{3}\right) + \tau C_4(\mathbf{x}) Y_{\frac{2}{3}}\left(\tfrac{2A\tau^{3/2}}{3}\right) , \tag{16}$$

with C_3 and C_4 are arbitrary spatial functions. The asymptotic behavior of this solution can be obtained by expanding around $\tau = 0$ to give, up to second order in τ

$$\phi^{(2)} \approx \frac{F(\mathbf{x})}{A^2} + C_3(\mathbf{x})\left(\frac{\Gamma(\frac{2}{3})}{A^{\frac{2}{3}}\pi}\right) + C_4(\mathbf{x})\tau^2 \ . \tag{17}$$

Now, assuming standard second order initial conditions [12] $\phi^{(2)}(\tau_0, \mathbf{x}) = 0$ and $\phi^{(2)\prime}(\tau_0, \mathbf{x}) = 0$ we find

$$C_3(\mathbf{x}) = \frac{-F/A^2}{(\frac{B_1}{B_2} - B_2)\tau_0^2 - B_1} \ , \quad C_4(\mathbf{x}) = \frac{-F/A^2}{1 + (\frac{B_2}{B_1} - 1)\tau_0^2} \ , \tag{18}$$

where $B_1 = \Gamma(\frac{2}{3})/(A^{\frac{2}{3}}\pi)$ and $B_2 = 3^{\frac{1}{6}}A^{\frac{2}{3}}/(6\Gamma(\frac{2}{3}))$. Therefore, $\phi^{(2)}$ asymptotically approaches a constant value in time which depends on τ_0, C_1, C_2, $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Furthermore, from the momentum and energy constraints, we obtain up to second order in τ

$$\chi^{(2)\alpha}_{\beta,\alpha}(\tau, \mathbf{x}) \approx Q_\beta(\mathbf{x}) \ , \quad \chi^{(2)\alpha\beta}_{,\alpha\beta}(\tau, \mathbf{x}) \approx G(\mathbf{x}) \ , \tag{19}$$

where Q_β and G are spatial functions given in [11]. Finally, substituting (19) in the asymptotic evolution equation for the second order trace-free perturbations we obtain

$$\chi^{(2)\prime\prime}_{\alpha\beta} - \frac{2}{\tau}\chi^{(2)\prime}_{\alpha\beta} + k^2\chi^{(2)}_{\alpha\beta} = S_{\alpha\beta}(k) + O(\tau) \ , \tag{20}$$

where $S_{\alpha\beta}$ is a function of k given in [11], and, as F above, is quadratic in first order quantities. This equation can be solved to give

$$\begin{aligned}\chi^{(2)}_{\alpha\beta} = \frac{S_{\alpha\beta}(k)}{k} &+ c_{\alpha\beta}(k\tau\cos k\tau - \sin k\tau) + \\ &d_{\alpha\beta}(k\tau\sin k\tau + \cos k\tau) \ ,\end{aligned} \tag{21}$$

where $c_{\alpha\beta}$ and $d_{\alpha\beta}$ are functions of k, which can be expanded around $\tau = 0$ and formally integrated to give

$$\chi^{(2)}_{\alpha\beta}(\tau, \mathbf{x}) \approx \int_0^\infty \left(\frac{S_{\alpha\beta}(k)}{k} + d_{\alpha\beta}\left(1 - \frac{k^2\tau^2}{2}\right)\right) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{k} \ . \tag{22}$$

As a result, as $\tau \to 0$, $\chi^{(2)}_{\alpha\beta}$ tends to a constant in time which depends on the asymptotic values of $\phi^{(2)}$ and $d_{\alpha\beta}$. These values are related to the first order initial free data by setting the initial conditions $\chi^{(2)}_{\alpha\beta}(\tau_0, k) = 0$ and $\chi^{(2)\prime}_{\alpha\beta}(\tau_0, k) = 0$, which result in

$$c_{\alpha\beta}(k) = \frac{-S_{\alpha\beta}\tau_0}{\frac{k^4\tau_0^4}{6} - k^3\tau_0^2} \ , \quad d_{\alpha\beta}(k) = \frac{-S_{\alpha\beta}}{1 + \frac{k\tau_0^2}{6}} \ . \tag{23}$$

So at second order there are still four independent quantities, namely $C_1, C_2, a_{\alpha\beta}$ and $b_{\alpha\beta}$, which correspond to the first order free initial data. Second order density perturbations can be computed once the metric is given [11].

In this way, we have shown that the second order scalar and tensor perturbations asymptotically approach constant values in time that are dependent on the initial conditions. This demonstrates that the asymptotic behavior of the first order perturbations is stable with respect to the second order nonlinear perturbations.

Now as was the case for the first order perturbations (see [5,6]), these asymptotic constant values cannot be removed globally by a gauge transformation. However, this can be done locally: by rescaling the time coordinate, one can write the asymptotic form of (1) for the case of the second order perturbations as

$$ds^2 = -dt^2 + e^{2\sqrt{\frac{\Lambda}{3}}t} A_{\alpha\beta}(x, y, z) dx^\alpha dx^\beta, \tag{24}$$

where $A_{\alpha\beta}$ is the asymptotic form of the spatial metric $\gamma_{\alpha\beta} \approx A_{\alpha\beta} + B_{\alpha\beta}\tau^2$, and both $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are spatial quantities corresponding to the sum of the first and second order asymptotic perturbation contributions. Then as shown in [6], there exists a coordinate change (valid inside the event horizon of a freely falling observer at $y^\alpha = 0$) given by

$$\begin{aligned} y^\alpha &= e^{Ht} x^\alpha \\ e^{HT} &= \frac{e^{Ht}}{\sqrt{1 - H^2 y^2}}, \end{aligned} \tag{25}$$

where $y^2 = y_1^2 + y_2^2 + y_3^2$, which allows the line element (24) to be asymptotically written as

$$ds^2 = -(1 - H^2 r^2) dT^2 + (1 - H^2 r^2)^{-1} dr^2 + r^2 d\Omega^2, \tag{26}$$

which represents the de Sitter solution in static coordinates [7]. In this sense, as in the case of first order perturbations, a freely falling observer will locally see the space asymptotically approach de Sitter in time.

This result is somehow unsatisfactory, as it relies on a specific frame and does not provide a direct insight into the fate of physical, observable gauge-invariant quantities [19] (see also [12,20]). Therefore, in order to shed more light on this behavior, we make use of curvature [18] and other invariants in order to characterize the asymptotic nature of the spacetime. The conformal electric and magnetic parts of the Weyl tensor can be written as

$$\begin{aligned} \tilde{E}^\alpha_\beta = a^2 E^\alpha_\beta &= \tilde{\theta}\tilde{\theta}^\alpha_\beta + \frac{a'}{a}\tilde{\theta}^\alpha_\beta + -\tilde{\theta}^\alpha_\gamma\tilde{\theta}^\gamma_\beta + \\ &\quad \frac{1}{3}\delta^\alpha_\beta(\tilde{\theta}^\mu_\nu\tilde{\theta}^\nu_\mu - \tilde{\theta}^2 - \frac{a'}{a}\tilde{\theta}) + \tilde{R}^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta\tilde{R} \ , \end{aligned} \tag{27}$$

$$\tilde{H}^\alpha_\beta = a^2 H^\alpha_\beta = \frac{1}{2}\gamma_{\beta\mu}(\eta^{\mu\gamma\delta}\sigma^\alpha_{\gamma;\delta} + \eta^{\alpha\gamma\delta}\sigma^\mu_{\gamma;\delta}) \ , \tag{28}$$

where $\tilde{\theta}^\alpha_\beta = a\theta^\alpha_\beta - \frac{a'}{a}\delta^\alpha_\beta$, $\tilde{R}^\alpha_\beta = a^2 R^{*\alpha}_\beta$, $\sigma^\alpha_\beta = \tilde{\theta}^\alpha_\beta - \frac{1}{3}\delta^\alpha_\beta\tilde{\theta}$ and $\eta^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma}/\sqrt{\gamma}$. It is then possible to show that asymptotically $\tilde{\theta}^\alpha_\beta \to 0$, $\tilde{R}^\alpha_\beta \to const$ (cf. [12] for the expression for $\tilde{R}^{*\alpha}_\beta$

perturbed to second order), $\tilde{H}^{\alpha}_{\beta} \to 0$ and $\tilde{E}^{\alpha}_{\beta} \to const$, and therefore we have $\theta^{\alpha}_{\beta} \to \sqrt{\Lambda/3}$, $\sigma^{\alpha}_{\beta} \to 0$, $R^{*\alpha}_{\beta} \to 0$, $E^{\alpha}_{\beta} \to 0$ and $H^{\alpha}_{\beta} \to 0$. These in turn give

$$\theta^2 \to \Lambda\ , \quad H^2 \to 0\ , \quad R \to 4\Lambda\ , \tag{29}$$

$$\sigma^2 \to 0\ , \quad E^2 \to 0\ , \quad R^* \to 0\ . \tag{30}$$

Now given that in the case of dust acceleration is zero and vorticity vanishes asymptotically, then Eqs. (29)-(30) are sufficient to prove invariantly that, locally, the nonlinearly perturbed spacetime approaches de Sitter.

To conclude, we have studied the asymptotic evolution of a dust plus Λ spacetime with vanishing vorticity, by considering second order perturbations of a dust plus Λ FLRW universe which is asymptotically de Sitter. We have proved the local asymptotic stability of the latter, using curvature and other spacetime invariants. This is a general second order gauge-invariant result for the case of irrotational dust with a positive cosmological constant, as guaranteed by the vanishing of σ^{α}_{β}, $R^{*\alpha}_{\beta}$, E^{α}_{β} and H^{α}_{β} in the background *and* (asymptotically) at first order [12,19,20]. It is well known that vorticity decays in an expanding perfect fluid with equation of state $p = w\rho$, $w < 2/3$, and one can prove that first order scalar perturbations decay for $-1 < w < 2/3$ in an asymptotically de Sitter background [11]. Gravitational wave perturbations are affected by the equation of state only through the background [15], and therefore their asymptotic evolution and their contribution to invariants is as described here. Therefore our results can in this sense be said to be more general.

We have demonstrated the validity of the cosmic no-hair conjecture in the nonlinear inhomogeneous settings considered here. This, together with previous exact and perturbative results, also supports the picture that, starting from inhomogeneous initial conditions, the universe as a whole may consist of homogeneous isotropic patches that have emerged from inflationary phases, and others that have undergone recollapse.

Non-linearities in the relativistic cosmology of the early and late universe have recently been investigated in a number of settings (see e.g. [21,12] and Refs. therein). We believe that such non-linear approximate methods are of great potential value in tackling many interesting problems in this field.

FCM thanks M. MacCallum for interesting comments, CMAT, U.Minho for support, FCT (Portugal) for grant PRAXIS XXI BD/16012/98, and the Relativity and Cosmology group at Portsmouth for warm hospitality. MB thanks Bruce Bassett for a useful remark and MB and RT would like to thank the 'Mathematical cosmology programme', Erwin Schrödinger Institute (Vienna) for hospitality while this work was finalised.

REFERENCES

[1] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977); S. W. Hawking and I. G. Moss, Phys. Lett. **1108** 35 (1982).

[2] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).

[3] A. A. Starobinsky, JETP Lett. **37**, 55 (1983); J. D. Barrow and J. Stein-Schabes, Phys. Lett. A **103**, 315 (1984); L. G. Jensen and J. A. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987); M. Bruni, S. Matarrese, and O. Pantano, Phys. Rev. Lett. **74**, 1916 (1995).

[4] A. A. Starobinsky, JETP Lett. **30**, 682 (1979).

[5] J. D. Barrow, in *The very early universe*, ed. by G. Gibbons, S. Hawking and S. Siklos (Cambridge University Press, Cambridge, 1983).

[6] W. Boucher and G. Gibbons, *ibidem.*

[7] Hawking S W & Ellis G F R, *The large scale structure of the space-time*, (Cambridge University Press, Cambridge, 1973)

[8] H. Friedrich, J. Geom. Phys. **3**, 101 (1986); A. A. Starobinsky and H.-J. Schmidt, Class. Quantum Grav. **4**, 695 (1987); J. D. Barrow and G. Gotz, Class. Quantum Grav. **6**, 1253 (1989); D. S. Goldwirth and T. Piran, Phys. Rep. **214**, 223 (1992); Y. Kitada and K. Maeda, Class. Quantum Grav. **10**, 703 (1993); K. Tomita and N. Deruelle, Phys. Rev. D **50**, 7216 (1994); J. Bicak and J. Podolsky, Phys. Rev. D **55**, 1985 (1997); S. Capozziello, R. de Ritis, A. A. Marino, GRG **30** 1247 (1998).

[9] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).

[10] A. R. Liddle and D. H. Lyth, *Cosmological inflation and Large-Scale structure* (Cambridge University Press, Cambridge, 2000).

[11] F. C. Mena, M. Bruni, R. Tavakol, (2001), *in preparation.*

[12] S. Matarrese, S. Mollerach and M. Bruni, Phys. Rev. D **58**, 043504 (1998).

[13] S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (1997).

[14] M. Bruni, P. K. S. Dunsby, and G. F. R. Ellis, Astrophys. J. **395**, 34 (1992).

[15] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[16] G. F. R. Ellis and M. Bruni, Phys. Rev. D, **40** 1804 (1989).

[17] A. Higuchi CQG **8**, 2005 (1991).

[18] J. Carminati and R. G. McLenaghan, J. Math. Phys. **32** 3135 (1991).

[19] M. Bruni and S. Sonego, Class. and Quantum Grav. **16**, L29 (1999).

[20] M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, Class. & Quantum Grav. **14**, 2585 (1997); S. Sonego and M. Bruni, Commun. Math. Phys. **193**, 209 (1998).

[21] B. A. Bassett, D. I. Kaiser, and R. Maartens, Phys. Lett. B **455**, 84 (1999); F. Finelli and R. Brandenberger, Phys. Rev. Lett. **82**, 1362 (1999); B. A. Bassett and F. Viniegra, Phys. Rev. D **62** 043507 (2000); A. R. Liddle, D. H. Lyth, K. A. Malik, and D. Wands, Phys. Rev. D **61**, 103509 (2000); Y. Nambu Phys. Rev. D **63**, 044013 (2001); N. Afshordi and R. Brandenberger, Phys. Rev. D **63**, 123505 (2001).